Transformaciones afines

La mejor manera de trabajar con la normal multivariada es vía matrices y vectores.

Sea $\tilde{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{2\times 1}$ y $\tilde{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{2\times 1}$

y $B_{2\times 2}$

$$\underline{\tilde{Y} = g(\tilde{X}) = \tilde{a} + B\tilde{X}} \leftarrow \text{transformaciones afines}$$

En este caso

$$\mathbb{E}(\tilde{Y}) = \tilde{a} + B\,\mathbb{E}(\tilde{X})$$

y $$\mathbb{E}(\tilde{X}) = \begin{pmatrix} \mathbb{E}(X_1) \\ \mathbb{E}(X_2) \end{pmatrix}$$

Además
$$\begin{aligned} \operatorname{Cov}(\tilde{Y}) &= \operatorname{Cov}(\tilde{a} + B\tilde{X}) \\ &= \operatorname{Cov}(B\tilde{X}) \\ &= B\operatorname{Cov}(\tilde{X})B^t \end{aligned}$$

$$\operatorname{Cov}(\tilde{X}) = \begin{pmatrix} \operatorname{Var}(X_1) & \operatorname{Cov}(X_1, X_2) \\ \operatorname{Cov}(X_2, X_1) & \operatorname{Var}(X_2) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_{21} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

Si $\quad \tilde{Y} = g(\tilde{X}) = \tilde{a} + B\tilde{X}$

$\Rightarrow \quad \tilde{Y} \sim ¿?$

$$f_{\tilde{Y}}(\tilde{y}) = |J|\, f_{\tilde{X}}(g^{-1}(\tilde{y}))$$

$$g^{-1}(\tilde{y}) = B^{-1}(\tilde{y} - \tilde{a})$$

$$\frac{d}{d\tilde{y}} g^{-1}(\tilde{y}) = B^{-1} \Rightarrow |J| = |\det(B^{-1})|$$

$$= \frac{1}{|\det(B)|}$$

Normal bivariada

Sea $X_1 \sim N(0,1)$ y $X_2 \sim N(0,1)$, con $Z_1 \perp Z_2$

$$\tilde{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Rightarrow \tilde{Y} = \tilde{\mu} + B\tilde{X}$$

$$g^{-1}(\tilde{Y}) = B^{-1}(\tilde{Y} - \mu)$$

$$f_{\tilde{X}}(\tilde{x}) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

$$= \frac{1}{2\pi} e^{-x_1^2/2} e^{-x_2^2/2} = \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2 + x_2^2)}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\tilde{x}^t \tilde{x}}$$

$$f_{\tilde{Y}}(\tilde{y}) = \frac{1}{2\pi} e^{-\frac{1}{2}[B^{-1}(\tilde{y}-\mu)]^t [B^{-1}(\tilde{y}-\mu)]} \frac{1}{|\det(B)|}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(\tilde{y}-\mu)^t (B^{-1})^t B^{-1}(\tilde{y}-\tilde{\mu})} \frac{1}{|\det(B)|}$$

$$(B^{-1})^t B^{-1} = (BB^t)^{-1}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(\tilde{y}-\mu)^t (BB^t)^{-1}(\tilde{y}-\mu)} \frac{1}{|\det(B)|}$$

$$BB^t = \Sigma \quad y \quad \det(\Sigma) = \det(B)^2$$

$$\Rightarrow \det(B) = \sqrt{\det(\Sigma)}$$

$$\Rightarrow f_{\tilde{Y}}(\tilde{y}) = \frac{1}{2\sigma_{11}\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}(\tilde{y}-\tilde{\mu})^t \Sigma^{-1} (\tilde{y}-\mu)}$$

$$\Rightarrow \mathbb{E}(\tilde{Y}) = \tilde{\mu}$$

$$Cov(\tilde{Y}) = B\, Cov(\tilde{X})\, B^t$$

con $Cov(\tilde{X}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= BB^t = \Sigma$$

$$\tilde{Y} \sim N(\tilde{\mu}, \Sigma)$$

$$\Rightarrow Cov(\tilde{Y}) = \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

en donde

$$Cov(X_1, X_2) = Cov(X_2, X_1) = \sigma_{12}$$

$$\rho = Corr(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2}$$

$\Rightarrow \operatorname{Cov}(X_1, X_2) = \rho\,\sigma_1\sigma_2$

$$\Rightarrow \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

Para la tarea

1) Hacer los cálculos y para

$$(X, Y) \sim N\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}\right)$$

obtener

$$\underline{\underline{f_{X,Y}(x,y)}}$$

2) obtener $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_{Y}(y)}$$

$= k\, h(y)$ ← una función de y

todo lo que no depende de y

$\boxed{x \text{ es una constante}}$

El modelo de regresión lineal simple

$$Y = \alpha + \beta X + \varepsilon, \quad \text{con} \quad \varepsilon \sim N(0, \sigma^2)$$

$$\mathbb{E}(Y \mid X = x) = \alpha + \beta x$$
$$Var(Y \mid X = x) = \sigma^2$$